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Combinatorial Reconstruction Problems

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A general technique for tackling various reconstruction problems is presented and applied to some old and some new instances of such problems.

1. INTRODUCTION

Let X be a (finite or infinite) set and let G be a (finite or infinite) group of automorphisms of X. Thus G acts on X and for every $g \in G$ the sequence $(gx)_{x \in X}$ is a permutation of X. For every subset Y of X and every $g \in G$, let gY be the set of all elements gy, for $y \in Y$. Clearly $|gY| = |Y|$ for every finite Y, and this defines an action of the group G on the power set of X. The *orbit* of a subset Y of X is, as usual, the set $Y^G = \{gY : g \in G\}$. We say that two subsets Y and Z of X are *G-equivalent* iff there is an element g of the group G mapping Y into Z, i.e., iff $Z \in Y^G$. Let R be a set of representatives of these orbits, i.e., a family consisting of a unique member of each orbit. For an integer $k \geqslant 1$ and for a subset Y of X of cardinality $m > k$, the *k-deck* of Y is the function $d = d_{Y,k} : R \to \{0, 1, 2, ...,\}$ where for each $r \in R$, $d(r)$ is the number of subsets of cardinality $m - k$ of Y which are G-equivalent to r.

Notice that $\sum_{r \in R} d(r) = \binom{m}{k}$ and that if Y and Z are G-equivalent then $d_{Y,k} = d_{Z,k}$, i.e., the k-deck of Y is determined by the orbit of Y under the action of G.

Several problems in combinatorics and in graph theory, known as reconstruction problems, are special cases of the following general problem:

Problem 1.1. Given a set X, a group G of automorphisms of X, and

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two integers m, k ($k < m$), decide if any subset Y of cardinality m of X is reconstructible (up to G-equivalence) from its k-deck.

The best known instance of this problem is the edge-reconstruction conjecture [H]. In this problem X is the set of all $\binom{n}{2}$ edges of a complete graph K_n on n labelled vertices and G is the group of all permutations of X induced by permuting the vertices of K_n. Every subset Y of X is a graph on n vertices, and its orbit is the set of all graphs isomorphic to it. Here $k = 1$, and the 1-deck is just the set of isomorphism types of all edge-deleted subgraphs of Y, each appearing according to its multiplicity. The edge-reconstruction conjecture asserts that for $k = 1$, $m \geqslant 4$, and X, G as above the answer to the decision Problem 1.1 is "yes," i.e., every graph with $m \geqslant 4$ edges can be reconstructed from its edge-deleted subgraphs.

The well-known result of Muller [M], extending that of Lovász [L], is that this is true for $m > 1 + \log_2(n!)$. More generally, it is shown in [GKR] that for X and G as above, for $k \geqslant 1$, and for $m > k + \log_2(n!)$ or for $m \geqslant \frac{1}{2}[\binom{n}{2} + k]$ the answer is also "yes."

Here we consider the general Problem 1.1. For a method that works in various other situations see [KR1, KR2]. Our results have several applications to various old and new reconstruction problems. In particular, we generalize a result of Nash–Williams on edge reconstruction of graphs to more general combinatorial structures.

The paper is organized as follows. In Section 2 we state and prove our main results concerning Problem 1.1. In Section 3 we consider some applications of these results to several reconstruction problems. In particular, we show that any colored cycle with $m > 5$ vertices of color 1 and $n - m$ vertices of color 2 is uniquely determined by the set of the m (unlabelled) colored cycles obtained from it by replacing, in all possible ways, a vertex of color 1 by a vertex of color 2. We also obtain several new results for the edge–reconstruction conjecture (for graphs and hypergraphs) and for the vertex-reconstruction conjecture. We further discuss some geometrical reconstruction problems. In particular we show that every polygon with $m > 8$ vertices in the plane is determined (up to isometry) by the set of isometry types of its vertex-deleted subpolygons. In the final Section 4 a few open problems are mentioned.

2. The Main Tools

In this section and throughout the paper we use the following standard notation from permutation group theory. For a permutation group G acting on X, and for Y and S subsets of X, let $Y^G = \{gY : g \in G\}$ denote the orbit of Y under the action of G, let $G_Y = \{g \in G : gY = Y\}$ denote the stabilizer of Y, and let $Y^{G_S} = \{gY : g \in G, gS = S\}$ denote the restricted orbit of Y under the action of G_S.

We start with the following somewhat technical but useful theorem.

THEOREM 2.1. *Let X be a (finite or infinite) set and let G be a (finite or infinite) group of automorphisms of X. Let Y be a subset of cardinality m of X, which is not reconstructible from its k-deck. Suppose, further, that there is a subset S of Y of cardinality $|S| = t$, with a finite stabilizer $|G_S| < \infty$, where $m - k \geqslant t$. Then there is a set T of cardinality $|T| \geqslant m - k + 1$, satisfying $S \subseteq T \subseteq Y$, and there is an $\varepsilon \in \{0, 1\}$ such that for every set K satisfying $S \subseteq K \subseteq T$ and $|K| \equiv \varepsilon (\mathrm{mod}\ 2)$, there is a $g \in G$ so that $T \cap gY = K$.*

Proof. Since Y is not reconstructible from its k-deck there is another subset $Y' \subseteq X$, $Y' \notin Y^G$ having the same k-deck as Y (and hence, clearly, $|Y'| = m$). For any set A containing S, let $f_1(A)$ denote the number of members of Y^G that contain A. (Notice that the condition $|G_S| < \infty$ easily implies that $f_1(S) < \infty$ and hence, also, $f_1(A) < \infty$ for all $S \subseteq A$.) Similarly, let $f_2(A)$ denote the number of members of Y'^G that contain A. Define also $f = f_1 - f_2$. One can easily check that for every set A containing S whose cardinality is at most $m - k$, $f_1(A)$ can be determined from the k-deck of Y. (In fact: $f_1(A) = (1/\binom{m-|A|}{k}) \sum_{r \in d_{Y,k}} |\{g \in G : A \subset gr\}|$.) Since $f_2(A)$ is determined from $d_{Y',k} = d_{Y,k}$ in the same manner we conclude that

$$f(A) = f_1(A) - f_2(A) = 0 \qquad \text{for all} \quad S \subseteq A,\ |A| \leqslant m - k. \tag{2.1}$$

Let T be a minimal set (with respect to containment), subject to the conditions $S \subseteq T \subseteq Y$ and $f(T) \neq 0$. Clearly, there is such a T, as $f(Y) = |G_Y| \geqslant 1 \neq 0$. Also, by (2.1), $|T| \geqslant m - k + 1$. Put $f(T) = f_1(T) - f_2(T) = b$.

Let K be a set satisfying $S \subseteq K \subseteq T$. By a simple inclusion–exclusion on the number of sets gY containing K we obtain

$$|\{g \in G : gY \cap T = K\}| = \sum_{K' : K \subseteq K' \subseteq T} (-1)^{|K' - K|} f_1(K').$$

Similarly

$$|\{g \in G : gY' \cap T = K\}| = \sum_{K' : K \subseteq K' \subseteq T} (-1)^{|K' - K|} f_2(K').$$

By subtracting the second equality from the first we conclude

$$|\{g \in G : gY \cap T = K\}| - |\{g \in G : gY' \cap T = K\}|$$
$$= \sum_{K' : K \subseteq K' \subseteq T} (-1)^{|K' - K|} f(K') = (-1)^{|T - K|} \cdot b,$$

where the last equality follows from the minimality in the choice of T.

We conclude that there is an $\varepsilon \in \{0, 1\}$ such that for every $S \subseteq K \subseteq T$ with $|K| \equiv \varepsilon (\mathrm{mod}\, 2)$ there is a $g \in G$ so that $gY \cap T = K$. (And also, for every $S \subseteq K \subseteq T$ with $|K| \equiv (1-\varepsilon)(\mathrm{mod}\, 2)$ there is a $g \in G$ so that $gY' \cap T = K$.)

This completes the proof. ▮

Remark 2.2. An equivalent formulation of the last proof is related to the study of null q-designs and to the Reed–Muller codes (see [AL, FP, MS). A real function h defined on the power set of a set X is a *null q-design* if for every $A \subseteq X$ of cardinality $|A| \leqslant q$ the equality $\sum_{B;\, A \subset B} h(B) = 0$ holds. One can easily check that if in Theorem 2.1, $t = 0$, then the function h defined on the power set of X by $h(A) = 1$ if $A \in Y^G$, $h(A) = -1$ if $A \in Y'^G$, and $h(A) = 0$ otherwise is a null $(m-k)$-design. In fact, Theorem 2.1 for this case can be derived from the main result of [FP]. Null q-designs have recently been generalized to arbitrary ranked finite lattices in [Le] where the ideas from [FP] are extended using the theory of Möbius functions. It is possible to obtain a version of Theorem 2.1 that deals with general ranked finite lattices, but for our purposes here the present formulation suffices.

Remark 2.3. Theorem 2.1 with X being the set of edges of the complete graph K_n on n vertices, G being the group of its automorphisms, $S = \varnothing$, $t = 0$, and $k = 1$ is just the result of Nash-Williams [N] concerning the edge reconstruction conjecture.

COROLLARY 2.4. *Let G be a finite group acting on X and let Y be a subset of cardinality m of X. If $2^{m-k} > |G|/|G_Y|$ then Y is reconstructible from its k-deck. In particular, if $2^{m-k} > |G|$ then every subset of cardinality m of X is reconstructible from its k-deck.*

Proof. Clearly $|Y^G| = |G|/|G_Y|$. If Y is not reconstructible from its k-deck, then, by Theorem 2.1 (with $S = \varnothing$) there is a subset T of cardinality $|T| \geqslant m-k+1$ of Y that has at least $2^{|T|-1} \geqslant 2^{m-k}$ distinct intersections with members of Y^G. Thus $|G|/|G_Y| \geqslant 2^{m-k}$. This contradicts the assumption and hence Y is reconstructible, as required. ▮

COROLLARY 2.5. *Let G be a group acting on X and suppose $Y \subseteq X$, $|Y| = m$. Suppose, further, that $S \subseteq Y$, $|S| = t$, $|G_S| < \infty$.*
If $2^{m-k-t} > |Y^{G_s}| \cdot \binom{m}{t}$ then Y is reconstructible from its k-deck.

Proof. Put $|Y^{G_S}| = s$. Suppose the assertion of the corollary is false and Y is not reconstructible from its k-deck. By Theorem 2.1 there is a subset T, $S \subseteq T \subseteq Y$, of cardinality $|T| \geqslant m-k+1$ and an $\varepsilon \in \{0, 1\}$ such that for every K, $S \subseteq K \subseteq T$, $|K| \equiv \varepsilon\ (\mathrm{mod}\ 2)$, there is a $g \in G$ so that $T \cap gY = K$. Put $O = \{gY : g \in G,\ S \subset gY\}$. Let $Y(t)$ be the set of all subsets of cardinality t of Y. Clearly $|Y(t)| = \binom{m}{t}$ and $O = \bigcup_{Z \in Y(t)} \{gY: g \in G, S = gZ\}$. One can easily check that for each fixed Z, $|\{gY;\ g \in G,\ S = gZ\}| \leqslant |Y^{G_S}| = s$. Hence $|O| \leqslant s\binom{m}{t}$. On the other hand, $T \backslash S$ has at least $2^{|T|-t-1} \geqslant 2^{m-k-t}$ distinct intersections with members of O. Thus $2^{m-k-t} \leqslant |O| \leqslant s\binom{m}{t}$, contradicting the assumption of the corollary. This completes the proof. ∎

3. Applications

3.1. *The Edge Reconstruction Conjecture for Graphs and Hypergraphs*

Let X be the set of all $\binom{n}{2}$ edges of the complete graph K_n on n labelled vertices and let G be the group of all automorphisms of K_n. As mentioned above, the edge reconstruction conjecture for graphs asserts that for these X and G, the answer to the decision Problem 1.1 is "yes" for all $m \geqslant 4$; i.e., every graph with n vertices and $m \geqslant 4$ edges is reconstructible from its edge-deleted subgraphs.

Corollary 2.5 (with $t = 1$) gives that if $2^{m-2} > m \cdot 2 \cdot (n-2)!$ then any graph with n vertices and m edges is reconstructible from its edge-deleted subgraphs. This is (slightly) better than Muller's result [M], which follows, of course, from Corollary 2.4.

Another result which follows from Theorem 2.1 is the following.

Proposition 3.1. *Let H be a connected graph with n vertices, m edges, and maximum degree Δ. If*

$$2^{m-k-n+1} > n \cdot \Delta!\, (\Delta - 1)^{n-\Delta-1} \tag{3.1}$$

then H is reconstructible from the collection of its k-edge deleted subgraphs.

We note that this implies that if $d \geqslant 2 + 2\log_2 \Delta + 2(k-1)/n$, where d is the average degree in H, then H is reconstructible from its k-edge deleted subgraphs. This extends a theorem of Caunter and Nash-Williams (cf. [B]), who proved the case $k = 1$ of the above statement.

Proof. Suppose the proposition is false and let H be a counterexample. Let Y be the set of edges of H and let S be the set of edges of an arbitrary spanning tree with maximum degree Δ in H. Since Y is not reconstructible from its k-deck, Theorem 2.1 implies that there is a set of edges T, $S \subseteq T \subseteq Y$, $|T| \geqslant m - k + 1$, and an $\varepsilon \in \{0, 1\}$, such that for every K, $S \subseteq K \subseteq T$, $|K| \equiv \varepsilon \pmod 2$, there is an automorphism g of K_n such that $gY \cap T = K$. Hence $|\{gY: S \subset gY\}| \geqslant 2^{|T|-|S|-1} \geqslant 2^{m-k-n+1}$. However, the set $\{gY: S \subset gY\}$ is the number of distinct copies of the graph H that contain the edges of the fixed spanning tree S. It is not too difficult to show that this number does not exceed $n \cdot \Delta!(\Delta-1)^{n-\Delta-1}$ (see, e.g., [B]). Hence, $2^{m-k-n+1} \leqslant n\Delta!(\Delta-1)^{n-\Delta-1}$, contradicting inequality (3.1). Therefore, H is reconstructible and the assertion of Proposition 3.1 holds. ▮

Similar results hold for uniform or non-uniform hypergraphs. Here X is the set of all 2^n edges of the complete hypergraph H_n on n labelled vertices and G is the group of $n!$ automorphisms of H_n. Corollary 2.4 implies here that any hypergraph with n vertices and m edges is reconstructible from its edge-deleted subgraphs, provided $2^{m-1} > n!$. Analogous results for reconstruction from *k-edge-deleted* subgraphs can be obviously formulated and proved, as well. Let us mention an application of the last result to the vertex reconstruction conjecture, that asserts that any graph with $n \geqslant 3$ vertices is reconstructible from its vertex-deleted subgraphs. This conjecture is not a special case of Problem 1.1, but for certain graphs it is. Indeed, let $G=(V, E)$ be a *2-connected* bipartite graph with classes of vertices A and B, where $|B| \geqslant 1+\log_2(|A|!)$ and in which each two vertices in B have distinct sets of neighbours. Associate each such graph G with a hypergraph $H(G)$ on the set of vertices A whose $|B|$ edges are the sets $e_b=\{a \in A;$ $ab \in E\}$ $(b \in B)$. One can easily check that for every $b \in B$, $H(G \backslash b)=$ $H(G)-e_b$ and hence, the vertex-reconstructibility of G follows from the edge-reconstructibility of the hypergraph $H(G)$.

3.2. *The Cycle Problem and Its Extension*

Let X be a cycle of length n and let Y be a coloring of its vertices by two colors, denoted 0 and 1. Suppose m vertices are colored 1 and suppose $1 \leqslant k < m$. If we are given the collection of all $\binom{m}{k}$(unlabelled) colored cycles obtained from Y be replacing k 1's by 0's, can we reconstruct Y (up to rotation and reflection)? This problem is obviously a special case of the general Problem 1.1 presented in Section 1. Here X is the set of n vertices of the cycle and G is the group of its automorphisms, i.e., the dihedral group of order $2n$. By Corollary 2.4, if $m > \log_2(2n)+k$ then Y is reconstructible. However, Corollary 2.5 is more effective here. Clearly, for every $x \in X$, $|G_x|=2$. Hence, by Corollary 2.5 if

$$m > \log_2(2m)+k+1 \tag{3.2}$$

then Y is reconstructible. For $k=1$ this shows that Y is reconstructible for all $m \geqslant 6$.

We note that for $k=m-2$ there are non-reconstructible colorings with larger values of m. To see this recall that a difference set modulo n is a subset $D \subseteq Z_n$ so that every $0 \neq x \in Z_n$ can be expressed uniquely as a difference $d-d'$ where $d, d' \in D$. Let D_1 and D_2 be two difference sets modulo n, so that $D_1 \neq D_2+x$ and $D_1 \neq -D_2+x$ for every $x \in Z_n$. Put $|D_1|=|D_2|=m$. Identify the n vertices of the cycle X with the elements of

Z_n and define two colorings Y_1 and Y_2 of it by $Y_i(x)=1$ iff $x \in D_i$ $(i=1, 2)$. One can easily check that these two colorings are not isomorphic, but the two collections of all $\binom{m}{2}$ colorings obtained from each of them by replacing in all possible ways $m-2$ 1's by 0's are identical. As an example take $n=13$, $m=4$, $k=2$, $D_1=\{0, 1, 4, 6\}$, and $D_2=\{0, 2, 8, 12\}$. This shows that for $m=4$ and $k=2$ not every Y is reconstructible. By inequality (3.2) for $k=2$ if $m \geqslant 7$ then Y is reconstructible.

The cycle problem can be generalized to arbitrary graphs. Let H be a graph on a set X of vertices, and let Y be a coloring of its vertices by the two colors 0 and 1, with m vertices colored 1. Suppose we are given the collection of all $\binom{m}{k}$ (unlabelled) colorings obtained from Y by replacing in all possible ways k 1's by 0's. Can we then reconstruct Y? Here G is the group Aut H of automorphisms of H. Thus, e.g., if $2^{m-k} > |\text{Aut } H|$ then Y is reconstructible by Corollary 2.4. The cycle problem is an interesting example, since here, by Corollary 2.5, our results do not depend on the length n of the cycle. For every fixed k, any coloring with sufficiently large m is reconstructible. There are many other examples of such graphs H, e.g., the product of an *n-cycle* with an edge, or with a triangle. Similar results hold for edge coloring, where the edge reconstruction conjecture is an instance of this last problem, with $H=K_n$ and $k=1$.

3.3. *Geometrical Reconstructions*

In the previous applications, the automorphism group G considered was always finite. Here we mention some infinite examples. Let X be a subset of the 2-dimensional Euclidean space $\mathbb{R}^2$ (we will be interested in the cases $X=R^2$ or $X=S^1 \equiv \{(x_1, x_2) \in R^2 : x_1^2+x_2^2=1\}$). Let G be the group of all isometries of X. Thus, for $X=\mathbb{R}^2$ or $X=S^1$, two subsets of X are G-equivalent iff they are isometric. If Y is a finite subset of cardinality m of X and $k<m$, then the *k-deck* of Y is the set of all $\binom{m}{k}$ isometry types of all the subsets of cardinality $m-k$ of Y, each appearing according to its multiplicity. We first consider the case $X=S^1$. Any finite subset of X is the set of vertices of a polygon, inscribed in a unit circle. Clearly, here, for every $x \in X$ $|G_x|=2$ and hence, by Corollary 2.5, if $m>k+1+\log_2(2m)$ then Y is reconstructible from its *k-deck*. Notice that the example given in the previous section is an example of a set of four points which is not reconstructible from its *2-deck*. For subsets of S^1 we can show that "similar" *k-decks* correspond to "similar" subsets. More formally, let us define the *distance* $d(Y, Y')$ between two finite subsets of the same cardinality Y and Y' of S^1 as

$$d(Y, Y')=\inf\left\{\sum_{y \in Y} \text{dist}(gy, f(y)) : g \in G, f\colon Y \to Y' \text{ a bijection}\right\}$$

where $\text{dist}(gy, f(y))$ is the usual (Euclidean) distance between gy and $f(y)$.

In this notation we prove:

THEOREM 3.1. *Suppose* Y, $Y' \subset S^1$, $|Y| = |Y'| = m > k + 1 + \log_2(2m)$, *and suppose that there is a bijection* h *from the set* F *of all* $\binom{m}{k}$ *subsets of cardinality* $m-k$ *of* Y *to the set* F' *of all* $\binom{m}{k}$ *subsets of cardinality* $m-k$ *of* Y' *so that for every* $T \in F$, $d(T, h(T)) < \varepsilon < 1/20^{2m+1}$. *Then* $d(Y, Y') < 40 \cdot m \cdot \varepsilon$.

Proof. The points of S^1 are just the points $\{e^{2\pi i x} : 0 \leqslant x < 1\}$. By the well-known result of Kronecker on simultaneous approximations (see, e.g., [HW]) there is an integer p, $1/20\varepsilon \leqslant p \leqslant 1/10\varepsilon$, such that for every point $y \in Y \cup Y'$ there is an integer j so that $\operatorname{dist}(y, e^{2\pi i j/p}) \leqslant (20\varepsilon)^{1/2m}/p < \frac{1}{5}\operatorname{dist}(1, e^{2\pi i/p})$.

Let $\bar{Y}$ be the set obtained from Y by replacing each of its points by the closest point of the form $e^{2\pi i j/p}$, and let $\overline{Y'}$ be the set obtained from Y' in the same manner. We claim that $d_{k,\bar{Y}} = d_{k,\overline{Y'}}$. Indeed, the correspondence h between F and F' gives a correspondence $\bar{h}$ between the set $\bar{F}$ of subsets of cardinality $m-k$ of $\bar{Y}$ to the corresponding set $\overline{F'}$ for $\overline{Y'}$, so that $d(\bar{T}, \bar{h}(\bar{T})) < \operatorname{dist}(1, e^{2\pi i/p})$ for all $\bar{T} \in \bar{F}$. However, one can easily check that this is possible iff $\bar{T}$ is isometric to $\bar{h}(\bar{T})$ for each such $\bar{T}$, since both are subsets of the discrete set of points $\{e^{2\pi i j/p} : 0 \leqslant j < p\}$. Therefore $d_{k,\bar{Y}} = d_{k,\overline{Y'}}$ and hence $\bar{Y}$ is isometric to $\overline{Y'}$. This clearly implies that $d(Y, Y') < 40m\varepsilon$, as needed. ∎

Next we consider the case $X = \mathbb{R}^2$. Here, for every subset S of two points in the plane there are four isometries g of $\mathbb{R}^2$ that map S to itself. Therefore, by Corollary 2.5 (with $t = 2$ and $s = 4$) if $2^{m-k-2} > 4\binom{m}{2}$ then any subset of m points in the plane is reconstructible from its k-deck. We note that by being more careful one can show that for $k = 1$ the inequality $2^{m-3} > 4m$ is enough, as the maximaum distance between a pair of m points in the plane is obtained at most m times (see, e.g., [E]).

Thus, every set of $m > 8$ points in $\mathbb{R}^2$ is uniquely determined (up to isometry) by the isometry types of all its point-deleted subsets. For some related results, see [KR3].

The above results can be generalized to sets of points in general position in higher dimensions. We omit the details.

4. Open Problems

The most interesting problem is of course the edge reconstruction conjecture. It does not seem that the general approach considered here will be very fruitful in tackling this problem; one will probably have to use more specific combinatorial arguments.

In the cycle problem, considered in Section 3, it would be nice to decide precisely for every pair m and k if every cycle coloring with m 1's can be reconstructed from the collection of the $\binom{m}{k}$ colorings obtained from it by replacing k 1's by 0's in all possible ways. Our results suggest that the size n of the colored cycle is not crucial here. A similar question arises naturally for the geometric problem. It would be interesting to decide, for every m and k, if every set of m points in $\mathbb{R}^2$ is determined (up to isometry) by its k-deck.

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